

BULLETIN
DE
L'ACADÉMIE POLONAISE
DES SCIENCES

Rédacteur en chef
K. KURATOWSKI

Rédacteur en chef suppléant
S. KULCZYŃSKI

CLASSE TROISIÈME

Rédacteur de la Série
L. INFELD

Comité de Rédaction de la Série

K. BORSUK, S. LESZCZYCKI, J. SAMSONOWICZ, M. ŚMIAŁOWSKI

VOLUME II
NUMÉRO 1

VARSOVIE 1954

PRINTED IN POLAND

PAŃSTWOWE WYDAWNICTWO NAUKOWE — WARSZAWA 1953

Nakład 1016 + 100 egz.	Rękopis dostarczono 1. XII. 1953
Ark. wyd. 3'2, ark. druk. 3	Podpisano do druku 5. III. 1954
Format B 5, 70×100 cm	Druk ukończono 15. III. 1954
Papier bezdrzewny sat. 80 g, kl. III	Zam. prod. 635/53 Cena zł 5,—

KRAKOWSKA DRUKARNIA NAUKOWA, KRAKÓW, UL. CZAPSKICH 4

Problème non linéaire d'Hilbert pour le système de fonctions

par

W. POGORZELSKI

Présenté par K. KURATOWSKI à la séance du 22 Juin, 1953

Soient, dans le plan de la variable complexe, p lignes fermées de Jordan L_1, L_2, \dots, L_p n'ayant pas de points communs et embrassant les domaines disjoints $S_1^-, S_2^-, \dots, S_p^-$, respectivement. Soit, en outre, L_0 une ligne fermée de Jordan embrassant toutes les lignes L_1, L_2, \dots, L_p et ne les coupant pas. Désignons par S_0^- le domaine non borné situé à l'extérieur de la ligne L_0 et par S^+ le domaine limité par les lignes $L_0, L_1, L_2, \dots, L_p$. Le problème aux limites d'Hilbert consiste à déterminer une fonction de la variable complexe $\Phi(z)$, holomorphe à l'intérieur des domaines $S^+, S_0^-, S_1^-, \dots, S_p^-$ séparément et dont les valeurs limites satisfassent à la relation linéaire donnée

$$(1) \quad \Phi^+(t) = G(t)\Phi^-(t) + g(t),$$

où $\Phi^+(t)$ désigne la valeur limite relative au domaine S^+ de la fonction $\Phi(z)$ au point t de la frontière $L = L_0 + \dots + L_p$ et $\Phi^-(t)$ désigne la valeur limite relative au domaine $S = S_0^- + S_1^- + \dots + S_p^-$ de la fonction $\Phi(z)$ au point t de la frontière L . Les fonctions $G(t)$ et $g(t)$ sont données en tout point t de L . Ce problème a été résolu dans une suite de travaux de T. Carleman, J. Privalov, F. Gachov et B. Hvedelidzé.

Nous appelons problème non linéaire d'Hilbert pour le système de fonctions celui de déterminer un système de fonctions de la variable complexe $\Phi_1(z), \Phi_2(z), \dots, \Phi_m(z)$ dont chacune est holomorphe dans les domaines S^+, S_0^-, \dots, S_p^- séparément et dont les valeurs limites $\Phi_\alpha^+(t), \Phi_\alpha^-(t)$ satisfont en tout point t de l'ensemble L aux relations:

$$(2) \quad \Phi_\alpha^+(t) = G_\alpha(t)\Phi_\alpha^-(t) + \lambda F_\alpha(t, \Phi_1^+(t), \dots, \Phi_m^+(t), \Phi_1^-(t), \dots, \Phi_m^-(t)), \quad (\alpha = 1, 2, \dots, m)$$

où $F_\alpha(t, u, \dots, u_{2m})$ sont les fonctions données de $2m+1$ arguments et λ est un paramètre. Nous supposons que chacune des fonctions données $G_\alpha(t)$ est définie dans l'ensemble des bandes fermées $\bar{Q}_0, \bar{Q}_1, \dots, \bar{Q}_p$ comprenant les lignes correspondantes L_0, \dots, L_p , qu'elle est holomorphe à l'intérieur de

toute bande *séparément* et que l'on a toujours $G_\alpha(t) \neq 0$. Nous supposons aussi que chacune des fonctions données $F_\alpha(z, u_1, \dots, u_{2m})$ est définie en tout point z situé à l'intérieur de chacune des bandes $\bar{\Omega}_0, \bar{\Omega}_1, \dots, \bar{\Omega}_p$ et aux points u_1, \dots, u_{2m} dans les cercles $|u_\nu| \leq R$. Nous supposons enfin que ces fonctions sont holomorphes dans chacune des bandes $\bar{\Omega}_0, \bar{\Omega}_1, \dots, \bar{\Omega}_p$ séparément.

D'après les résultats de Gachov et Hvedelidzé concernant le problème linéaire d'Hilbert, nous pouvons affirmer que toute solution (2) du problème considéré s'exprime par les formules

$$(3) \quad \Phi_\alpha(z) = \frac{\lambda}{2\pi i} X_\alpha(z) \int_L \frac{F_\alpha[\tau, \varphi, (\tau), \dots, \varphi_{2m}(\tau)]}{X_\alpha^+(\tau)(\tau - z)} d\tau + X_\alpha(z) P_\alpha(z),$$

$X_\alpha(z)$ étant la solution dite *canonique* du problème homogène d'Hilbert (c'est-à-dire satisfaisant à la condition limite $X_\alpha^+(t) = G_\alpha(t) X_\alpha^-(t)$ et ayant le plus petit ordre à l'infini), $P_\alpha(z)$ étant une fonction entière arbitraire et $\varphi_1(t), \dots, \varphi_{2m}(t)$ formant une solution du système d'équations intégrales

$$(4) \quad \varphi_\alpha(t) = \lambda \bar{F}_\alpha^*[t, \varphi_1(t), \dots, \varphi_{2m}(t)] + \lambda \int_L \frac{\bar{F}_\alpha^{**}[t, \tau, \varphi_1(\tau), \dots, \varphi_{2m}(\tau)]}{\tau - t} d\tau, \\ (\alpha = 1, 2, \dots, 2m)$$

où les intégrales ont la valeur principale de Cauchy, les fonctions \bar{F}_α^* et \bar{F}_α^{**} sont définies et holomorphes dans les bandes $\Omega_0, \Omega_1, \dots, \Omega_p$, comprenant les lignes L_0, \dots, L_p , et dans les cercles $|u_\nu| \leq R$.

Pour résoudre le système d'équations intégrales (4) non linéaires et à haute singularité, considérons dans chacune des bandes Ω_ν ($\nu = 0, 1, \dots, p$) une suite de lignes fermées $\{A_\nu^n\}$ telles que A_ν^n embrasse A_ν^{n+1} et qui convergent vers la ligne limite A_ν embrassant la ligne L_ν ; considérons de même dans chacune des bandes Ω_ν une suite de lignes fermées $\{B_\nu^n\}$ telles que B_ν^{n+1} embrasse B_ν^n et qui convergent vers la ligne limite B_ν embrassée par L_ν . Formons maintenant $2m$ suites de fonctions

$$(5) \quad \varphi_\alpha^1(z), \varphi_\alpha^2(z), \dots, \varphi_\alpha^n(z), \dots \quad (\alpha = 1, 2, \dots, 2m)$$

à l'aide des relations de récurrence

$$\varphi_\alpha^{n+1}(z) = \lambda \bar{F}_\alpha^*[z, \varphi_1^n(z), \dots, \varphi_{2m}^n(z)] + \frac{1}{2} \lambda \int_{A^{n+1} + B^{n+1}} \frac{\bar{F}_\alpha^{**}[z, \tau, \varphi_1^n(\tau), \dots, \varphi_{2m}^n(\tau)]}{\tau - z} d\tau$$

A^{n+1} et B^{n+1} désignant l'ensemble des lignes A_ν^{n+1} et B_ν^{n+1} .

En se basant sur le théorème de résidu et sur la propriété de l'intégrale principale de Cauchy, on peut montrer que les suites (5) convergent

vers les fonctions limites $\varphi_\alpha(z)$, holomorphes dans les bandes comprises entre les lignes A_ν et B_ν , le module du paramètre λ étant suffisamment petit. Ces fonctions limites satisfont au système d'équations intégrales (4) en tout point des lignes L_ν et fournissent la solution du problème non linéaire d'Hilbert dans sa forme (3).

Le travail contenant un développement plus détaillé du problème non linéaire d'Hilbert paraîtra sous le même titre dans les *Annales Polonici Mathematici* (1954).

Problème du mouvement stationnaire dans une couche gazeuse rayonnante

par

W. POGORZELSKI

Présenté par K. KURATOWSKI à la séance du 22 Juin, 1953

Nous allons établir dans cette communication l'existence du mouvement stationnaire dans une couche gazeuse en tenant compte du rayonnement polychromatique intérieur. Nous supposons que la couche est comprise entre deux plans $x=0$ et $x=a$, et que la pression $p(x)$, la densité $\varrho(x)$, et la température $T(x)$ ne dépendent que de la coordonnée x . Nous supposons en outre que la vitesse et l'intensité du champ de gravitation sont perpendiculaires aux plans limites et que leurs valeurs $u(x)$ et $g(x)$ dépendent seulement de la coordonnée x .

D'après l'hydrodynamique et la théorie du rayonnement, les fonctions $u(x)$, $\varrho(x)$, $T(x)$ et $p(x)$ à l'état stationnaire doivent satisfaire dans l'intervalle $(0, a)$ au système d'équations

$$\begin{aligned} u \frac{du}{dx} &= -\frac{1}{\varrho} \frac{dp}{dx} + g(x) \\ \frac{d(\varrho u)}{dx} &= 0, \quad p = R\varrho T \\ (1) \quad c\varrho u \frac{dT}{dx} + p \frac{du}{dx} &= 2\pi \int_0^\pi \int_0^\infty \alpha X_n(\lambda, \Theta) \sin \Theta d\lambda d\Theta - 4\pi \int_0^\infty \alpha J(\lambda, T) d\lambda. \end{aligned}$$

$X_n(\lambda, \Theta)$ désigne ici l'intensité du rayonnement de longueur d'onde λ au point $M(x)$ dans une direction qui fait l'angle Θ avec l'axe Ox ; α désigne le coefficient d'absorption et $J(\lambda, T)$ la fonction universelle de Planck; enfin c désigne la chaleur spécifique du gaz. Il est supposé que $\alpha = \nu(\lambda)\varrho(x)$, la fonction $\nu(\lambda)$ de la longueur d'onde étant donnée.

Nous admettons que les intensités du rayonnement $X_1(\lambda, \Theta)$ et $X_2(\lambda, \Theta)$ aux surfaces de la couche $x=0$ et $x=a$ sont les fonctions données a priori, de même que les valeurs $u_0 \neq 0$, $\varrho_0 > 0$, et $T_0 > 0$ des fonctions $u(x)$, $\varrho(x)$

et $T(x)$ pour $x=0$. Nous en concluons que notre problème est amené à la recherche des fonctions $T(x)$ et $\varrho(x)$ satisfaisant dans l'intervalle $(0, a)$ au système d'équations intégrales non linéaires

$$(2) \quad \begin{aligned} T(x) &= T_0 + \int_0^x \frac{R u_0 \varrho_0 g(y) T(y) \varrho^2(y) + [R T(y) \varrho^3(y) - u_0^2 \varrho_0^2] \hat{\Omega}(y, T, \varrho)}{R(R+c) u_0 \varrho_0 T(y) \varrho^2(y) - c u_0^3 \varrho_0^3} dy \\ \varrho(x) &= \varrho_0 + \int_0^x \frac{c u_0 \varrho_0 g(y) \varrho^3(y) - R \varrho^3(y) \hat{\Omega}(y, T, \varrho)}{R(R+c) u_0 \varrho_0 T(y) \varrho^2(y) - c u_0^3 \varrho_0^3} dy. \end{aligned}$$

$\hat{\Omega}$ désigne ici l'opération fonctionnelle suivante sur les fonctions T, ϱ

$$(3) \quad \begin{aligned} \hat{\Omega}(x, T, \varrho) &= f_1 \left[\int_0^x \varrho(s) ds \right] + f_2 \left[\int_x^a \varrho(s) ds \right] \\ &+ \int_0^a \Phi \left[T(x'), \left| \int_x^{x'} \varrho(s) ds \right| \right] \varrho(x') dx' - F[T(x)], \end{aligned}$$

où les fonctions F, f_1, f_2 et Φ sont déterminées par les formules

$$(4) \quad \begin{aligned} F(T) &= 4\pi \int_0^\infty \nu(\lambda) J(\lambda, T) d\lambda, \\ f_i(w) &= \int_0^\infty \int_0^{\frac{\pi}{2}} X_i(\lambda, \Theta) \nu(\lambda) e^{-\frac{\nu(\lambda)}{\cos \Theta} w} \sin \Theta d\Theta d\lambda. \quad (i=1, 2) \\ \Phi(T, w) &= \int_0^\infty \int_0^{\frac{\pi}{2}} \nu^2(\lambda) J(\lambda, T) e^{-\frac{\nu(\lambda)}{\cos \Theta} w} \operatorname{tg} \Theta d\Theta d\lambda. \end{aligned}$$

Les fonctions F, f_i, Φ sont ainsi définies pour toute valeur positive de T et w . On démontre que la fonction Φ et les dérivées des fonctions f_i tendent à l'infini comme $|\log w|$ avec $w \rightarrow 0$; il en est de même de la dérivée Φ'_T .

La valeur absolue de la dérivée Φ'_w croît comme $1/w$ avec $w \rightarrow 0$. En nous basant sur ces propriétés des fonctions (4) et en supposant que $R(R+c)T_0 - cu_0^2 \neq 0$, on peut appliquer la méthode des approximations successives au système (2). On constate ainsi l'existence d'une solution $T(x)$, $\varrho(x)$ l'épaisseur de la couche a étant suffisamment petite.

Le travail contenant un développement plus détaillé du problème de la couche paraîtra sous le même titre dans les *Annales Polonici Mathematici* (1954).

Sur une propriété analytique des homéomorphismes définies sur des continus plans

par

K. KURATOWSKI

Présenté à la séance du 16 Novembre, 1953

Désignons par \mathcal{S}_2 le plan des nombres complexes, le point à l'infini y inclus („plan de Gauss"). Soit A un sous-ensemble fermé de \mathcal{S}_2 et soit

$$(1) \quad \mathcal{S}_2 - A = R_0 + R_1 + \dots$$

la décomposition de l'ensemble $\mathcal{S}_2 - A$ en régions-composantes (finie ou infinie).

Soit

$$(2) \quad p_j \in R_j \quad \text{où} \quad j=0, 1, \dots$$

Soit enfin f une transformation continue de l'ensemble A à valeurs complexes différentes de 0 et de ∞ :

$$(3) \quad w = f(z) \quad \text{où} \quad 0 \neq w \neq \infty.$$

D'après un théorème général (dû à S. Eilenberg, voir [1] et [2], p. 393), il existe une fonction continue u , définie sur \mathcal{S}_2 , et une suite d'entiers k_0, k_1, \dots qui s'annulent à partir d'un certain indice et tels qu'on a

$$(4) \quad k_0 + k_1 + \dots = 0,$$

$$(5) \quad f(z) = e^{u(z)} \cdot (z - p_0)^{k_0} \cdot (z - p_1)^{k_1} \cdot \dots \quad \text{pour} \quad z \in A^*.$$

La fonction f étant fixe, on peut évidemment admettre que les composantes de l'ensemble $\mathcal{S}_2 - A$ ont été numérotées de façon que

$$(6) \quad k_j \neq 0 \quad \text{pour} \quad j \leq n \quad \text{et} \quad k_i = 0 \quad \text{pour} \quad i > n$$

(sauf le cas, bien entendu, où tous les exposants k_j s'annulent).

On a donc à la place de (4) la formule

$$(7) \quad k_0 + \dots + k_n = 0.$$

*) Nous convenons que $z - \infty \equiv 1$.

L'identité (5) sera écrite sous la forme d'équivalence *):

$$(8) \quad f(z) \sim (z - p_0)^{k_0} \dots (z - p_n)^{k_n}.$$

Les exposants k_0, k_1, \dots étant indépendants du choix des points p_0, p_1, \dots (pourvu que la formule (2) soit satisfaite), nous avons donc attaché à toute fonction f une suite bien déterminée d'entiers k_0, \dots, k_n .

La fonction f a été supposée jusqu'ici une fonction continue arbitraire (satisfaisant à la condition (3)). Admettons à présent qu'elle soit une homéomorphie. Nous nous proposons de résoudre le problème suivant: trouver une condition imposée au système k_0, \dots, k_n qui soit nécessaire et suffisante pour l'existence d'un continu localement connexe A^{**} et d'une transformation homéomorphe f de A satisfaisant aux formules (2), (3), (6) — (8).

Nous allons démontrer que telle est la condition

$$(9) \quad |k_0| + \dots + |k_n| \leq 2n.$$

Nous allons établir, en effet, le théorème suivant:

Théorème. *Toute transformation homéomorphe f d'un continu localement connexe A assujettie aux conditions (1) — (3) et (6) — (8) satisfait à l'inégalité (9).*

Inversement — comme nous allons démontrer ailleurs [3] — étant donnés un système de points (distincts) p_0, \dots, p_n et un système d'entiers k_0, \dots, k_n satisfaisant aux conditions (6), (7) et (9), il existe un continu localement connexe A qui coupe le plan entre tout couple p_i, p_j (pour $i \neq j$) ainsi qu'une homéomorphie f définie sur A et satisfaisant à la formule (8) ***).

I. Préliminaires. Nous allons nous servir des théorèmes suivants:

1. Soit $f(z) \sim (z - q_0)^{s_0} \dots (z - q_r)^{s_r}$, où les points q_0, \dots, q_r appartiennent à R_m , tandis que $q_i \in S_2 - A - R_m$ pour $i > r$. On a alors

$$k_m = s_0 + \dots + s_r.$$

Voir [2], p. 393, 2.

2. Soit f une homéomorphie définie sur une courbe simple fermée C qui coupe le plan S_2 entre les points p et q . On a alors

$$f(z) \sim (z - p)^k (z - q)^{-k} \quad \text{où} \quad |k| \leq 1.$$

Pour le cas où la courbe C ne contient pas le point à l'infini, voir la démonstration donnée dans [2], p. 432, 1.

Le cas où $\infty \in C$ se ramène à celui-ci à l'aide de l'homographie

$$w = h(z) = \frac{z - p}{z - q}.$$

*) Cette équivalence signifie (cf. [2], p. 388) que ses membres sont homotopes relativement au plan euclidien diminué du point 0.

**) Le continu A admet donc une représentation paramétrique continue $z = z(t)$ sur l'intervalle $0 \leq t \leq 1$.

***) Pour des généralisations des énoncés ci-dessus, voir [3].

En effet, la fonction $g(w) = fh^{-1}(w)$ est une homéomorphie définie sur la courbe simple fermée $h(C)$ qui coupe \mathcal{S}_2 entre 0 et ∞ . Elle est donc de la forme

$$g(w) = e^{u(w)} \cdot w^k, \quad \text{où} \quad |k| \leq 1, \quad \text{pour} \quad w \in h(C).$$

On a par conséquent

$$f(z) = gh(z) = e^{uh(z)} \cdot (z - p)^k (z - q)^{-k} \quad \text{pour} \quad z \in C.$$

3. Le continu A contient une double suite de courbes simples fermées C_{ij} qui coupent le plan \mathcal{S}_2 entre p_i et p_j .

Voir [2], p. 361, 5.

II. Démonstration du théorème. Il y a deux cas à distinguer:

1°. Pour tout indice j , sauf, peut-être, pour un seul (l'indice n , par exemple), l'ensemble A contient une courbe simple fermée K_j qui coupe le plan entre le point p_j et tous les autres points du système p_0, \dots, p_n . On a alors d'après I, 1

$$f(z) \sim (z - p_j)^{k_j} \cdot (z - p_n)^{k_0 + \dots + k_{j-1} + k_{j+1} + \dots + k_n},$$

pour $z \in K_j$. Donc, d'après I, 2, $|k_j| = 1$ pour $0 \leq j < n$.

Il en résulte d'après (7) que $|k_n| \leq n$.

L'inégalité (9) s'en déduit aussitôt.

2°. Dans le cas contraire, il existe deux indices, 0 et n par exemple, pour lesquels la courbe $K = C_{0,n}$ (cf. I, 3) coupe le plan en deux régions D et E telles qu'on ait (en numérotant de façon convenable les régions R_0, \dots, R_n):

$$(10) \quad p_0, \dots, p_r \in D, \quad p_{r+1}, \dots, p_n \in E,$$

où

$$(11) \quad 1 \leq r \leq n - 2.$$

Posons: $M = A \cdot \bar{D}$ et $N = A \cdot \bar{E}$. Il vient

$$M + N = A \quad \text{et} \quad MN = K,$$

de sorte que la somme et le produit des ensembles (fermés) M et N sont des continus localement connexes. Il en résulte (cf. [2], pp. 83,5 et 164, 10) que les ensembles M et N , eux aussi, sont des continus localement connexes. De plus, comme on le voit facilement, R_0, \dots, R_r sont les composantes de $\mathcal{S}_2 - M$ qui contiennent respectivement les points p_0, \dots, p_r et E en est la composante qui contient les points p_{r+1}, \dots, p_n . On en conclut d'après (8) et I, 1 que

$$(12) \quad f(z) \sim (z - p_0)^{k_0} \cdot \dots \cdot (z - p_r)^{k_r} \cdot (z - p_n)^{k_{r+1} + \dots + k_n} \quad \text{pour} \quad z \in M$$

et d'une façon analogue

$$(13) \quad f(z) \sim (z - p_0)^{k_0 + \dots + k_r} \cdot (z - p_{r+1})^{k_{r+1}} \cdot \dots \cdot (z - p_n)^{k_n} \quad \text{pour} \quad z \in N.$$

Enfin, d'après I, 3, on a pour $z \in K$

$$(14) \quad f(z) \sim (z - p_0)^k (z - p_n)^{-k}$$

où $|k| \leq 1$. Il vient d'après (10), (13) et I, 1

$$(15) \quad k_0 + \dots + k_r = k \text{ et } k_{r+1} + \dots + k_n = -k.$$

Procédons à présent par induction. Le théorème étant vrai pour $n = 1$ (d'après 1°), admettons qu'il soit vrai pour tout $s < n$. Le nombre des facteurs dans (12) et dans (13) étant $\leq n$ (d'après (11)), on en conclut que: 1° dans le cas où $|k| = 1$, on a

$$(16) \quad |k_0| + \dots + |k_r| + 1 \leq 2(r+1)$$

et

$$(17) \quad 1 + |k_{r+1}| + \dots + |k_n| \leq 2(n-r),$$

2° dans le cas où $k = 0$, on a

$$(18) \quad |k_0| + \dots + |k_r| \leq 2r$$

et

$$(19) \quad |k_{r+1}| + \dots + |k_n| \leq 2(n-r-1).$$

Dans les deux cas, l'inégalité (9) se trouve réalisée.

Institut Mathématique de l'Académie Polonaise des Sciences

OUVRAGES CITÉS

- [1] Eilenberg S., *Transformations continues en circonférence et la topologie du plan*, Fund. Math. **26** (1936), 61—112.
- [2] Kuratowski K., *Topologie II*, Monogr. Mat. **21** (1950).
- [3] Kuratowski K., Fund. Math. **41** (à paraître).

Les espaces métriques des suites limitables par les méthodes continues

par

L. WŁODARSKI

Présenté par S. MAZUR à la séance du 18 Mai, 1953

S. Mazur et W. Orlicz [3] ont introduit l'espace B_0 dans les recherches sur les ensembles de suites limitables par les méthodes de Toeplitz [1] et [2]. Cette idée peut être appliquée avec succès aux méthodes fonctionnelles, pourvu qu'on se borne à une classe particulière de ces méthodes. On parvient ainsi aux résultats suivants (dont les démonstrations complètes paraîtront dans *Studia Math.* vol. 14).

Soit $\{a_n(t)\}$ une suite de fonctions continues dans l'intervalle $0 \leq t < T \leq \infty$. Cette suite détermine une méthode A de limitation. On dit que la suite $x = \{\xi_n\}$ est limitable par la méthode fonctionnelle A au nombre ξ , lorsque $\lim_{t \rightarrow T-} \sum_{n=0}^{\infty} a_n(t) \xi_n = \xi$; en symbole: $A\text{-}\lim \xi_n = \xi$.

L'ensemble A^* des suites limitables par la méthode A est dit le domaine de A . Deux méthodes sont appelées équivalentes lorsqu'elles ont les mêmes domaines et sont compatibles. La fonction $A(t, x) = \sum_{n=0}^{\infty} a_n(t) \xi_n$ s'appelle la transformée de x . Une méthode fonctionnelle A est dite continue, lorsque:

- 1° les $a_n(t)$ sont continues pour $0 \leq t < T$ ($n = 0, 1, 2, \dots$);
- 2° il existe une suite croissante $\{\tau_m\} (\tau_0 = 0)$ convergente vers T et telle que la convergence de la série $A(t, x)$ pour $t = \tau_{m-1}$ et pour $t = \tau_m$ entraîne sa convergence uniforme dans l'intervalle $\tau_{m-1} \leq t \leq \tau_m$, quels que soient ξ_n .

Il résulte de cette définition que la transformée, si elle existe, est continue et que la série $A(t, x)$ est convergente uniformément dans tout l'intervalle $0 \leq t \leq t_0 < T$.

Les méthodes d'Abel et de Borel sont continues. Toute méthode de limitation de Toeplitz donnée par la matrice $(a_{m,n})$ équivaut à une méthode continue, à savoir:

$$a_n(t) = \frac{t_{m+1} - t}{t_{m+1} - t_m} a_{m,n} + \frac{t - t_m}{t_{m+1} - t_m} a_{m+1,n}$$

pour $t_m \leq t \leq t_{m+1}$, où $\{t_m\}$ est une suite arbitraire telle que $t_m \rightarrow T$.

Si $f(t) = \sum_{n=0}^{\infty} a_n t^n$ pour $|t| < R$, $f(t) \neq 0$ et $\lim_{t \rightarrow R-} \frac{t_n}{f(t)} = 0$ ($n = 0, 1, 2, \dots$), la méthode F ayant la transformée

$$F(t, x) = \frac{1}{f(t)} \sum_{n=0}^{\infty} a_n t^n \xi_n$$

est dite méthode de puissance.

Les méthodes de puissance constituent une sous-classe très importante des méthodes continues. Pour $\{\tau_m\}$ on peut prendre une suite arbitraire qui tend à R .

Une méthode est dite permanente lorsque chaque suite convergente est limitable vers sa limite au sens habituel.

Théorème I. *Pour qu'une méthode continue soit permanente, il faut et il suffit que:*

$$1^\circ \sum_{n=0}^{\infty} |a_n(t)| \leq K < \infty \text{ pour } 0 \leq t < T,$$

où K ne dépend pas de t ;

$$2^\circ \lim_{t \rightarrow T-} a_n(t) = 0 \quad (n = 0, 1, 2, \dots);$$

$$3^\circ \lim_{t \rightarrow T-} \sum_{n=0}^{\infty} a_n(t) = 1.$$

Toute méthode de puissance F , où $a_n \geq 0$, est permanente.

Théorème II. *Le domaine A d'une méthode continue A est un espace B_0 avec les pseudonormes:*

$$|x_n| = |\xi_n| \quad (n = 0, 1, 2, \dots), \quad |x|_{A,m} = \sup_n \left| \sum_{k=0}^n a_k(t_m) \xi_k \right| \quad (m = 1, 2, \dots),$$

$$|x|_A = \sup_{0 \leq t < T} \left| \sum_{n=0}^{\infty} a_n(t) \xi_n \right|,$$

où $\{t_m\}$ est une suite dense dans $(0, T)$ contenant la suite $\{\tau_m\}$ (voir la définition de la méthode continue).

Théorème III. *La forme générale des fonctionnelles dans le domaine A^* d'une méthode continue A est:*

$$f(x) = \int_0^T A(t, x) dh(t) + \sum_{n=0}^{\infty} c_n \xi_n,$$

où $h(t)$ est à variation bornée dans $\langle 0, T \rangle$, c_n sont des nombres arbitraires pour $0 \leq n \leq n_0$ et

$$c_n = \sum_{m=1}^{m_0} d_{m,n} a_n(t_m) \quad \text{pour } n > n_0 \quad (n_0 - \text{naturel}),$$

$d_{m,n}$ étant assujetti à la seule condition

$$\sum_{n=0}^{\infty} |d_{m,n} - d_{m,n+1}| < \infty \quad (m = 1, 2, \dots, m_0),$$

et $\{t_m\}$ étant des nombres définis dans le théorème II.

La méthode B est dite plus générale que A lorsque $A^* \subset B^*$, c'est-à-dire lorsque toute suite limitable A est limitable B .

Théorème IV. a. *Toute méthode fonctionnelle B qui est plus générale qu'une méthode continue A constitue dans A^* une fonctionnelle linéaire (c'est-à-dire additive et continue).*

b. Si A est une méthode continue, il existe, pour toute fonctionnelle linéaire $f(x)$ définie dans A^ , une méthode T de Toeplitz qui est plus générale que A et qui est compatible dans A avec $f(x)$.*

Une méthode A est dite parfaite au point $x_0 = \{\xi_n^0\}$ lorsque, pour toute méthode fonctionnelle B permanente et plus générale que A , on a $B\text{-lim } \xi_n^0 = A\text{-lim } \xi_n^0$.

Une méthode est dite parfaite lorsqu'elle est parfaite en tout point $x \in A^*$.

Théorème V. *Pour qu'une méthode continue et permanente A soit parfaite dans x_0 , il faut et il suffit que x_0 soit dans A un point d'accumulation d'éléments de l'espace (c) (c'est-à-dire de celui des suites convergentes).*

La méthode A est dite uniforme lorsque, pour toute suite $x = \{\xi_n\}$ limitable A vers zéro, la série $A(t, x)$ est uniformément convergente dans l'intervalle $0 \leq t < T$.

Théorème VI. *Toute méthode continue et uniforme est parfaite.*

Théorème VII. *Soit A une méthode continue et permanente, donnée par la suite de fonctions $\{a_n(t)\}$. Si $a_n(t) \geq 0$ pour $t \geq t_0$ ($n = 0, 1, 2, \dots$), la méthode A est parfaite pour toute suite $x = \{\xi_n\}$ telle que $\xi_n \geq A\text{-lim } \xi_n$ par $n \geq n_0$ ou $\xi_n \leq A\text{-lim } \xi_n$ pour $n \geq n_0$.*

Soient: e la suite dont chaque terme est le nombre 1 et e_n la suite dont le n -ième terme est 1 et tous les autres sont des zéros.

Théorème VIII. *Pour qu'une méthode A continue et permanente soit parfaite au point x_0 , il faut et il suffit que toute fonctionnelle linéaire, définie dans A^* et s'annulant pour les vecteurs e et e_n ($n = 0, 1, 2, \dots$) s'annule pour x_0 .*

Théorème IX. *Pour qu'une méthode continue et permanente A soit parfaite au point $x_0 = \{\xi_n^0\}$, il faut et il suffit que les égalités*

$$\int_0^{T-} a_k(t) d h(t) + c_k = 0 \quad (k = 1, 2, \dots),$$

entraînent l'égalité

$$\int_0^{T-} A(t, x_0) dh(t) + \sum_{k=0}^{\infty} c_k \xi_k^0 = 0$$

pour toute fonction $h(t)$ à variation bornée dans $\langle 0, T \rangle$ et pour toute suite $\{c_k\}$ satisfaisant aux conditions énoncées dans le théorème III.

Théorème X. Toute méthode continue et permanente A est parfaite dans l'ensemble des suites bornées (c'est-à-dire pour toute suite bornée $x_0 \in A^*$).

Théorème XI (commun avec C. Ryll-Nardzewski). Toute méthode de puissances F telle que $T < \infty$ et $a_n > 0$ est parfaite.

Corollaire. La méthode d'Abel est parfaite.

Une méthode de Toeplitz $A(a_{m,n})$ sera dite extraite d'une méthode continue $A\{a_n(t)\}$ si $a_{m,n} = a_n(t_m)$, où $\{t_m\}$ est une suite croissante de nombres positifs tels que $\lim_{m \rightarrow \infty} t_m = T$.

Théorème XII (commun avec C. Ryll-Nardzewski). Toute méthode de Toeplitz extraite d'une méthode de puissance F telle que $T < \infty$ et $a_n > 0$ est parfaite.

Une méthode A est dite translatrice lorsqu'elle jouit de la propriété suivante: si la suite $\{\xi_n\}$ est limitable A , la suite (qui en diffère par la suppression du premier terme ou par l'addition de l'avant-premier terme supplémentaire) est limitable A vers le même nombre.

Théorème XIII. Il existe des méthodes continues, permanentes et translatrices qui ne sont pas compatibles.

OUVRAGES CITÉS

- [1] Mazur S., *Eine Anwendung der Theorie der Operationen bei der Untersuchung der Toeplitzschen Limitierungsverfahren*, *Studia Math.* **2** (1930), 40.
- [2] Mazur S. et Orlicz W., *Sur les méthodes linéaires de sommation*, *Comptes Rendus* **196** (1933), 33.
- [3] Mazur S. et Orlicz W., *Sur les espaces métriques linéaires* (1), *Studia Math.* **10** (1948), 184.

A Theorem on Fixed Points

by

K. BORSUK

Communicated at the meeting of November 16, 1953

1. It is known (see, for example [2], p. 227) that every continuous mapping f of an acyclic, locally connected curve (also called a dendrite) into itself has a fixed point, i. e. a point a such that $f(a)=a$.

The purpose of this note is to prove the following, more general

Theorem 1. *Let A be an arcwise connected, acyclic curve. Every continuous mapping f of A into itself has a fixed point.*

A continuum C is said to be *unicoherent* if for every two continua C_1 and C_2 such that $C=C_1+C_2$ the set $C_1 \cdot C_2$ is a continuum. A continuum C is said to be *hereditarily unicoherent* if every one of its subcontinua is unicoherent. We assert that every acyclic curve is hereditarily unicoherent. In fact, if a curve A contains a not unicoherent continuum C , then there exists in C a 1-dimensional true cycle γ not homologous to zero in C . Since $\dim A=1$, the true cycle γ also is not homologous to zero in A (see [1]), hence A is not acyclic.

It follows that to obtain theorem 1 it is sufficient to prove the following

Theorem 2. *Every continuous mapping of an arcwise connected and hereditarily unicoherent curve into itself has a fixed point.*

2. Lemma. *Every subcontinuum of an arcwise connected and hereditarily unicoherent curve is arcwise connected.*

Proof. Let A_0 be a subcontinuum of an arcwise connected and hereditarily unicoherent curve A . Consider two points $a, b \in A_0$ and let L be a simple arc lying in A and having a and b as endpoints. Since A is hereditarily unicoherent, $A_0 \cdot L$ is connected. It follows by $a, b \in A_0 \cdot L \subset L$ that $L \subset A_0$ and consequently A_0 is arcwise connected.

Since in every indecomposable continuum C there exist two points which cannot be joined in C by any simple arc (see for instance [2] p. 150), we get

Corollary. *An arcwise connected, hereditarily unicoherent curve does not contain any indecomposable continuum.*

3. Lemma. Let A be an arcwise connected and hereditarily unicoherent curve and φ a one-to-one continuous function mapping the ray $<0, \infty)$ into A . Then the closure of the set $P = \varphi(<0, \infty))$ is a simple arc.

Proof. A point $a \in A$ will be said to be a *limit point* for φ , if there exists a sequence $\{x_n\} \subset <0, \infty)$ such that $\lim_{n \rightarrow \infty} x_n = \infty$ and $\lim_{n \rightarrow \infty} \varphi(x_n) = a$.

We assert that no single limit point a_0 for φ belongs to P . Otherwise there would exist an $x_0 \geq 0$ such that $a_0 = \varphi(x_0)$. The set $C = \varphi(<x_0, \infty))$ is a unicoherent curve and, by the lemma 2, it is arcwise connected. Moreover for every $x_1 > x_0$

$$C = \overline{\varphi(<x_1, \infty))} + \varphi(<x_0, x_1>).$$

Since C is unicoherent and $\varphi(x_0), \varphi(x_1) \in \overline{\varphi(<x_1, \infty))} \cdot \varphi(<x_0, x_1>)$ we conclude that $\varphi(<x_0, x_1>) \subset \varphi(<x_1, \infty))$. Hence

(1) for every $x_1 > x_0$ the set $\varphi(<x_1, \infty))$ is dense in C .

Since C is decomposable, there exist two continua C_0 and C_1 such that

(2) $C = C_0 + C_1$; $C - C_0 \neq 0 \neq C - C_1$; $a_0 = \varphi(x_0) \in C_0$.

Putting $G_v = C - C_v$ for $v = 0, 1$, we obtain two non-empty, open and disjoint subsets of C . It follows by (1) that the sets $H_v = \varphi(<x_0, \infty)) \cdot G_v$ for $v = 0, 1$, are non-empty, disjoint and open in $\varphi(<x_0, \infty))$.

By (1) and (2) the set $\varphi(<x_1, \infty))$, where $x_1 > x_0$, is not contained in any of the sets C_0 and C_1 . It follows that there exist two numbers x , such that $x_0 < x < y$ and that

$$\varphi(x) \in C_1, \quad \varphi(y) \in C_1 \quad \text{and} \quad \varphi((x, y)) \subset H_1.$$

We infer that the continuum $C_1 + \varphi(<x, y>) \subset A$ is not unicoherent, contrary to the hereditary unicoherence of A .

Thus we have shown that no single limit point for φ belongs to $\varphi(<0, \infty))$. It follows that for every $x > 0$ the sets $\varphi(<0, x>)$ and $\overline{\varphi(<x, \infty))}$ have only the point $\varphi(x)$ in common. Consequently $\varphi(x)$ cuts the set $\overline{\varphi(<0, \infty))}$ between the point $\varphi(0)$ and every point $a \in \overline{\varphi(<0, \infty))} - \varphi(<0, \infty))$. But, by the lemma of section 2, there exists a simple arc L joining in $\overline{\varphi(<0, \infty))}$ the points $\varphi(0)$ and a . It follows that $\varphi(<0, \infty)) \subset L$ and consequently $\varphi(<0, \infty))$ is a subarc of L .

4. Proof of theorem 2. Let A be an arcwise connected, hereditarily unicoherent curve. By the lemma of section 2, for every two points $a, b \in A$ ($a \neq b$) there exists in A exactly one simple arc \widetilde{ab} with the end points a and b . In the case $a = b$ we put $\widetilde{ab} = 0$. If $p \in \widetilde{ab} - (a) - (b)$, then we shall say that p lies between a and b , written $p \mu ab$. Evidently $p \mu ab$ implies $a \bar{\varepsilon} p \bar{\nu}$.

Suppose now that there exists a continuous mapping f of A into itself without fixed points. Then there exists a positive ε such that

3) $\varrho(p, f(p)) \geq \varepsilon$ for every $p \in A$.

Now let us assume that for a natural n there is constructed a system of n points $a_1, a_2, \dots, a_n \in A$ satisfying the following conditions:

$$(4_n) \quad \varrho(a_i, a_{i+1}) = \frac{1}{2}\varepsilon \quad \text{for every } i < n,$$

$$(5_n) \quad \text{if } p\mu a_i a_{i+1}, \text{ then } \varrho(a_i, p) < \frac{1}{2}\varepsilon \quad \text{for every natural } i < n,$$

$$(6_n) \quad \overline{a_1 a_n} = \sum_{i=1}^{n-1} \overline{a_i a_{i+1}}$$

$$(7_n) \quad a_n \mu a_1 f(a_n) \quad \text{if } n > 1.$$

Evidently the system containing only one point $a_1 \in A$ satisfies, for $n=1$, the conditions $(4_n) - (7_n)$. We assert that every system a_1, a_2, \dots, a_n satisfying $(4_n) - (7_n)$ can be completed by adding point $a_{n+1} \in A$ to the system a_1, a_2, \dots, a_{n+1} satisfying the conditions $(4_{n+1}) - (7_{n+1})$.

By (3) there exists a point a_{n+1} such that

$$(8) \quad a_{n+1} \mu a_n f(a_n),$$

$$(9) \quad \varrho(a_n, a_{n+1}) = \frac{1}{2}\varepsilon,$$

$$(10) \quad \text{if } p\mu a_n a_{n+1}, \text{ then } \varrho(a_n, p) < \frac{1}{2}\varepsilon.$$

Now conditions (4_{n+1}) and (5_{n+1}) are immediate consequences of conditions (4_n) and (9) and conditions (5_n) and (10) respectively. To establish condition (6_{n+1}) it suffices to show that

$$(11) \quad \overline{a_1 a_n} \cdot \overline{a_n a_{n+1}} = (a_n).$$

Otherwise there would exist a point $b \neq a_n$ such that $b \in \overline{a_1 a_n} \cdot \overline{a_n a_{n+1}}$. Then

$$(12) \quad \overline{a_1 a_{n+1}} \subset \overline{a_1 b} + \overline{b a_{n+1}} \subset \overline{a_1 a_n} + \overline{a_n a_{n+1}} = (a_n).$$

But $a_{n+1} \mu a_n f(a_n)$ implies that $a_n \bar{\in} \overline{a_{n+1} f(a_n)}$. It follows by (12) that the continuum $\overline{a_1 a_{n+1}} + \overline{a_{n+1} f(a_n)}$ lies in $A - (a_n)$. We infer that $\overline{a_1 f(a_n)} \subset A - (a_n)$, in contradiction to (7_n) .

From (11) and (8) we get

$$(13) \quad a_{n+1} \mu a_1 f(a_n).$$

Let us observe that (3) and (10) imply, for every $p \in \overline{a_n a_{n+1}}$:

$$\varrho(a_n, f(p)) \geq \varrho(p, f(p)) - \varrho(a_n, p) > \frac{1}{2}\varepsilon.$$

But $f(\overline{a_n} f(a_{n+1})) \subset \overline{f(a_n a_{n+1})}$. It follows by (9)

$$(14) \quad a_{n+1} \bar{\in} \overline{f(a_n) f(a_{n+1})}.$$

Suppose now that condition (7_{n+1}) does not hold. Then a_{n+1} does not lie between a_1 and $f(a_{n+1})$. But $a_{n+1} \neq f(a_{n+1})$ (by (3)) and $a_{n+1} \neq a_1$ (by (13)).

We conclude that $a_{n+1} \in \overline{a_1 f(a_{n+1})}$. It follows by (14) that the continuum $\overline{a_1 f(a_{n+1}) + f(a_n) f(a_{n+1})}$ lies in $A - (a_{n+1})$. Applying the lemma of section 2, we infer that $\overline{a_1 f(a_n)} \subset A - (a_{n+1})$, in contradiction to (13).

Thus we have shown (under the supposition that there exists a continuous mapping f of A into itself satisfying the condition (3)) that every system a_1, a_2, \dots, a_n satisfying (4_n)—(7_n) can be enlarged to a system $a_1, a_2, \dots, a_n, a_{n+1}$ satisfying the conditions (4_{n+1})—(7_{n+1}). It follows that there exists a sequence $a_1, a_2, \dots, a_n, \dots$ such that for every natural n the system a_1, a_2, \dots, a_n satisfies the conditions (4_n)—(7_n).

Let φ_n be a homeomorphism mapping the interval $\langle n-1, n \rangle$ onto $\overline{a_n a_{n+1}}$ in such a manner that $\varphi_n(n-1) = a_n$ and $\varphi_n(n) = a_{n+1}$. It follows by (6_n) that putting

$$\varphi(x) = \varphi_n(x) \quad \text{for } 1 \leq x \leq n$$

we get a one-to-one continuous mapping of the ray $\langle 0, \infty \rangle$ onto the set

$$P = \sum_{n=1}^{\infty} \overline{a_n a_{n+1}}.$$

By the lemma of section 3 we infer that \bar{P} is a simple arc. Hence there exists a homeomorphism h mapping \bar{P} onto the interval $\langle 0, 1 \rangle$. Evidently the sequence $\{h(a_n)\}$ is monotonous, hence convergent. Consequently also, the sequence $\{a_n\}$ should be convergent, which contradicts (9).

5. Theorem implies the following

Corollary. *Let A be a curve contractible in itself. Every continuous mapping of A into itself has a fixed point.*

REFERENCES

- [1] Alexandroff P. S., *Dimensionstheorie. Ein Beitrag zur Geometrie der abgeschlossenen Mengen*, Math. Ann. **106** (1932), 161—238.
- [2] Kuratowski K., *Topologie II*, Monografie Matematyczne XXI, Warszawa-Wrocław, 1950.

On the Existence of Linear Order in a Group

by

J. ŁOŚ

Communicated by K. KURATOWSKI at the meeting of November 16, 1953

A group G is called linearly ordered, if an ordering relation on G is given such that, for $a, b, x, y \in G$:

$$\text{if } a < b, \text{ then } xay < xby.$$

In my paper [1] I have shown that the class **L** of all groups which admit a linear order may be defined in a very simple way: there exists a sequence of general statements *) $\alpha_1, \alpha_2, \dots$ such that a group G belongs to **L** if, and only if, every α_i is true in G .

In this paper a direct proof of the theorem above is given and, with the help of an idea which is due to Masao Ohnishi [3], the sequence of statements which define the class **L** is constructed.

1. Each finite subset A of a group G is called a *segment* of G . Two segments $A_1 \subset G_1$ and $A_2 \subset G_2$ are *isomorphic*, if there exists a one-one mapping $\varphi(A_1) = A_2$, such that:

$$\begin{aligned} &\text{if } a, b, ab \in A_1, \quad \text{then } \varphi(ab) = \varphi(a)\varphi(b) \\ &\text{and if } c, d, cd \in A_2, \quad \text{then } \varphi^{-1}(cd) = \varphi^{-1}(c)\varphi^{-1}(d). \end{aligned}$$

2. A formula of the form $\sigma = \tau$, where σ and τ are words written in some of the variables $x_1, x_2, \dots, y_1, y_2, \dots$, is called an *equation*. With the help of propositional connectives ($+$ — disjunction, \cdot — conjunction, $'$ — negation) we form *open formulas* from equations. The statements equivalent to one of the forms

$$\prod_{x_{s_1}} \dots \prod_{x_{s_n}} \beta \quad \text{or} \quad \sum_{x_{s_1}} \dots \sum_{x_{s_n}} \beta,$$

where β is an open formula and \prod and \sum quantifiers, are called respectively *general* and *existential* statements. The negation of an existential statement is a general statement.

*) For the definition of general statements see § 2.

3. Each segment A_1 of a group G_1 may be described by an existential statement α (called its description) in such a way that α is true in an arbitrary group G_2 if, and only if, there exists a segment A_2 of G_2 isomorphic with A_1 . So, e.g. the segment (1, 2) of the multiplicative group of real numbers is described by the statement

$$\sum_{x_1} \sum_{x_2} (x_1 \neq x_2 \cdot x_1^2 = x_1 \cdot x_1 x_2 = x_2 \cdot x_2 x_1 = x_2 \cdot x_2^2 \neq x_1 \cdot x_2^2 \neq x_2) *).$$

Obviously there exists only a denumerable set of non-isomorphic segments; by $\alpha_1, \alpha_2, \dots$ we shall designate the sequence of all their descriptions.

4. A segment A of G is called *ordered* by an ordering relation $<$ on A , if from $a, b, x, y, xay, xby \in A$ and $a < b$ it follows that $xay < xby$. If two segments A_1 and A_2 of a group G are ordered respectively by the relations $<_1, <_2$ then the relations $<_1, <_2$ are called *consistent*, if for $a, b \in A_1 \cap A_2$, $a <_1 b$ is equivalent to $a <_2 b$. Obviously, if the segment $A_1 \cup A_2$ can be ordered, then A_1 and A_2 can be ordered in a consistent way.

With the help of the theorem on consistent choice ([2], p. 235), it may easily be proved that:

if each segment of a group G can be ordered, then the whole group G can be ordered.

The property that a segment can be ordered is evidently invariant with respect to isomorphism. Let $\alpha_{k_1}, \alpha_{k_2}, \dots$ be the sequence of descriptions of this segment which admit no order. It follows that a group G can be ordered if, and only if, each α_{k_i} is false in G and therefore:

the group G belongs to \mathbf{L} if, and only if, each statement $\alpha'_{k_1}, \alpha'_{k_2}, \dots$ is true in G .

Each α_{k_i} is existential, therefore each α'_{k_i} is a general statement. It can easily be shown that the sequence $\alpha_{k_1}, \alpha_{k_2}, \dots$ is infinite and cannot be reduced to a finite one.

5. It is known that a linear order in a group G is uniquely determined by the sub-semigroup of the "positive" elements and conversely: to each invariant sub-semigroup P which for each $a \in G$, $a \neq e$, contains either a or a^{-1} (and not both), there exists a linear order in G such that P is the set of positive elements.

For $a_1, \dots, a_n \in G$, let $[a_1, \dots, a_n]$ denote the invariant sub-semigroup generated by a_1, \dots, a_n . If $G \in \mathbf{L}$, then for each $a_1, \dots, a_n \in G$, different from e , there exist numbers $\varepsilon_1, \dots, \varepsilon_n$, ($\varepsilon_i = \pm 1$), such that $e \notin [a_1^{\varepsilon_1}, \dots, a_n^{\varepsilon_n}]$. It is sufficient to choose the numbers ε_i in such a manner, that $a_i^{\varepsilon_i}$ is positive. From the theorem on consistent choice it follows that the converse theorem is also true:

If for each $a_1, \dots, a_n \in G$, $a_i \neq e$, there exist numbers $\varepsilon_1, \dots, \varepsilon_n$, ($\varepsilon_i = \pm 1$) such that $e \notin [a_1^{\varepsilon_1}, \dots, a_n^{\varepsilon_n}]$, then $G \in \mathbf{L}$.

* $x_1 \neq x_2$ stands for $(x_1 = x_2)'$.

6. Let \mathcal{P} denote the least set of words $\sigma(x_{k_1}, \dots, x_{k_n}; y_{l_1}, \dots, y_{l_m})$ such that

all variables x_i belong to \mathcal{P} ;

if $\sigma_1, \sigma_2 \in \mathcal{P}$, then $\sigma_1 \sigma_2 \in \mathcal{P}$;

if $\sigma \in \mathcal{P}$, then $y_j \sigma y_j^{-1} \in \mathcal{P}$.

It is easy to see that

$a_0 \in [a_1, \dots, a_n]$ if, and only if, there exist

$$b_1, \dots, b_m \in G \quad \text{and} \quad \sigma(x_1, \dots, x_n; y_{l_1}, \dots, y_{l_m}) \in \mathcal{P},$$

such that

$$a_0 = \sigma(a_1, \dots, a_n; b_1, \dots, b_m).$$

Let $t_i = \langle \varepsilon_1^{(i)}, \dots, \varepsilon_n^{(i)} \rangle$, $i = 1, 2, \dots, 2^n$, be all n -tuplets in which $\varepsilon_j^{(i)} = \pm 1$.

By σ/t_i , where $\sigma = \sigma(x_1, \dots, x_n; y_{l_1}, \dots, y_{l_m}) \in \mathcal{P}$, the word

$$\sigma(x_1^{\varepsilon_1^{(i)}}, \dots, x_n^{\varepsilon_n^{(i)}}; y_{l_1}, \dots, y_{l_m})$$

will be denoted.

Now let $\alpha_1, \alpha_2, \dots$ be the sequence of all statements of the form

$$\prod_{x_1} \dots \prod_{x_n} \prod_{y_s} \dots \prod_{y_r} [(x_1 \neq e \dots x_n \neq e) \rightarrow (\sigma_1/t_1 \neq e + \dots + \sigma_{2^n}/t_{2^n} \neq e)]$$

where $\sigma_1, \dots, \sigma_{2^n} \in \mathcal{P}$ (n arbitrary).

By means of the theorem given in 5 it is easy to verify that $G \in \mathbf{L}$ if, and only if, all the statements $\alpha_1, \alpha_2, \dots$ are true in G .

Institute of Mathematics, Polish Academy of Sciences

REFERENCES

- [1] Łoś J., *On the Extension of Models*, to appear in *Fund. Math.*
- [2] — and Ryll-Nardzewski C., *On the Applications of Tychonoff's theorem in Mathematical Proofs*, *Fund. Math.*, **38** (1951), 233—237.
- [3] Ohnishi Masao, *Linear-Order on a Group*, *Osaka Math. Journal*, **4** (1952), 17—18.

Angular Distribution of Photo-Neutrons from Beryllium

by

H. NIEWODNICZAŃSKI and M. WIELOWIEJSKA

Communicated by H. NIEWODNICZAŃSKI at the meeting of June 22, 1953

Angular distribution of the photo-neutrons produced in beryllium by the γ -rays of ThD was determined by measuring the density of the proton recoil tracks in nuclear emulsions.

The earliest measurements of the angular distribution of photo-neutrons from beryllium were carried out in 1935 by Chadwick and Goldhaber [1]. They used γ -rays from a radon source and found a spherically symmetrical distribution of neutrons. Later, Goloborodko and Rosenkevich [2] using γ -rays from a radium source obtained a similar result. The effective γ -rays used in both these investigations were of several different energies owing to the complexity of the γ -ray spectrum from the radium family. Three different sources of monoenergetic γ -rays for determining the angular distribution of the photo-neutrons from ^9Be were used by Hamermesh, Hamermesh and Wattenberg [3]. The distribution was found to be spherically symmetrical for the 1.70 MeV and 1.81 MeV γ -rays. For the 2.76 MeV γ -rays from ^{24}Na , however, the distribution of the photo-neutrons was of the form $a + b \sin^2 \vartheta$ with $a/b = 1.22$, where ϑ is the angle between the direction of the photo-neutrons and that of the γ -ray beam.

These results were in general agreement with the theory of the photodisintegration of ^9Be elaborated by Guth and Mullin [4], [5]. According to this theory the angular distribution of the photo-neutrons from ^9Be for γ -ray energies just above the threshold value must be of the spherical symmetry, departing from that for higher γ -ray energies and obeying the distribution rule $a + b \sin^2 \vartheta$ with the ratio a/b decreasing with increasing γ -ray energy. In the light of this theory the results obtained by Chadwick and Goldhaber and by Goloborodko and Rosenkevich, may easily be explained since in the γ -ray spectrum of the radium family the energy of the most intensive effective γ -rays only very slightly exceeds the photodisintegration threshold value.

In all former investigations mentioned above the photoneutrons were detected by ionization chambers sensitive to the thermal neutrons only.

For that purpose the slowing down of photo-neutrons in a layer of paraffin wax was necessary. This may introduce some degree of uncertainty of results being obtained especially for non-monoenergetic γ -rays producing neutrons.

The method of detecting the photo-neutrons applied in our experiments was more direct. The experimental arrangement used in these experiments, represented in Fig. 1, was very similar to that of our previous work on

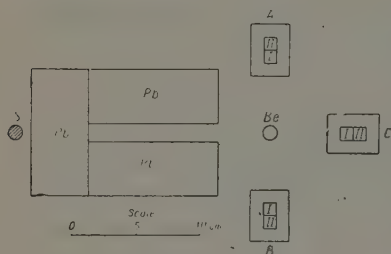


Fig. 1

the high energy spectrum of ThD [6]. Gamma rays from a mesothorium source (of about 35 mC. radium equivalent) filtered in a 4.5 cm. thick lead block, in order to diminish the soft γ -ray background in the photographic emulsion, passed through a cylindrical channel of a diameter of 1.5 cm. in a lead block to a piece of metallic beryllium situated in front of the opening of the lead channel. The shape of this metallic beryllium was only approximately spherical, of a diameter of about 1.1 cm. Neutrons produced in beryllium in the photo-reaction ${}^9\text{Be}(\gamma, n){}^8\text{Be}$, not being previously slowed down, also passed to the emulsion of three Ilford C2 100 μ nuclear emulsion plates. As shown in Fig. 1 the plate in position C was situated in front of the opening of the lead block channel in the direction of the γ -ray beam. The plates in positions A and B were placed on both sides of the beryllium piece at the same distance from it as the plate in position C, as shown in Fig. 1. The planes of each of three plates were inclined in respect to the plane of the horizontal projection of the arrangement shown in Fig. 1 at an equal angle of 13° . The whole arrangement was surrounded by a thick layer of lead not shown in Fig. 1.

As exposed in our previous work the γ -ray spectrum of ThD, effective in producing photo-neutrons in beryllium, consists mainly of two lines of the energy of 2.62 MeV and 3.20 MeV respectively. The maximum lengths of the proton recoil tracks (in the case of head-on collisions) in the emulsions of C2 plates are 12μ and 22μ , corresponding to two groups of photo-neutrons of the energy of 0.88 MeV and 1.35 MeV. These energy values of neutrons correspond to the mentioned energies of the γ -rays 2.62 MeV and 3.20 MeV if the most recent threshold value of 1.67 MeV for the reaction (γ, n) in ${}^9\text{Be}$ [7] and the recoil energy of the ${}^8\text{Be}$ nucleus are taken into account.

Proton recoil tracks were counted on each plate on equal areas of 2 cm^2 . These areas were situated at an equal distance from the centre of the piece of beryllium. The investigated areas of the plate in position C

were situated in the direction of the γ -ray beam ($\vartheta = 0^\circ$) and those of the plates in positions *A* and *B* at right angles to the γ -ray beam ($\vartheta = 90^\circ$).

Both the horizontal projection (parallel to the plane of plates) and the depth of all observed tracks were measured. In order to obtain more definite results only tracks forming an angle with the direction of the entering neutrons below 45° were taken into account. When elaborating these measurements the shrinkage of the C2 emulsion was admitted to be 2.5.

Three plates in positions *A*, *B* and *C* were exposed in each experiment simultaneously. On each plate two fields of an area of 1 cm.^2 each, denoted by I and II and situated as shown in Fig. 1 at the same distance from the centre of the beryllium piece, were elaborated. The results of proton track counting obtained for the two mentioned neutron groups in two experiments lasting 4 and 5 days respectively are given in Tables I and II.

TABLE I

Numbers of proton recoil tracks from photo-neutrons produced in ^9Be nuclei by 2.62 MeV γ -rays for fields I and II together

Experiment		Plate in pos. <i>A</i>	Plate in pos. <i>B</i>	Plates in pos. <i>A</i> and <i>B</i> mean number	Plate in pos. <i>C</i>
No.	days				
1.	4	379	373	376	236
2.	5	449	457	453	270
Total		—	—	829	506

TABLE II

Numbers of proton recoil tracks from photo-neutrons produced in ^9Be nuclei by 3.20 MeV γ -rays for fields I and II together

Experiment		Plate in pos. <i>A</i>	Plate in pos. <i>B</i>	Plates in pos. <i>A</i> and <i>B</i> mean number	Plate in pos. <i>C</i>
No.	days				
1.	4	53	56	54	29
2.	5	63	62	63	36
Total		—	—	117	65

In order to avoid errors caused by any possible asymmetry in geometrical arrangement of the experiment, mean values of the numbers of proton tracks counted on plates in positions *A* and *B* were taken. For the sake of checking how far the numbers of tracks in emulsions of plates in positions *A* and *B* are influenced by the spatial orientation of the beryllium neutron source of a non-spherical shape, the following experiment was performed. Four plates were exposed for 4 days consecutively in positions *A* and *B* at two opposite orientations of the piece of beryllium. The results obtained for field I are given in Table III.

TABLE III

Numbers of proton recoil tracks corresponding to the 2.62 MeV γ -ray photons obtained in 4 days on field I

	Plate in pos. A	Plate in pos. B
1-st position of the beryllium piece	220	220
2-nd position of the beryllium piece (opposite to the 1-st)	225	214

As seen in Table III the differences in numbers of counted tracks are within the limits of statistical deviations.

The numbers of neutrons of a given energy, reaching equal areas of a photographic plate, are proportional to the numbers of tracks of recoil

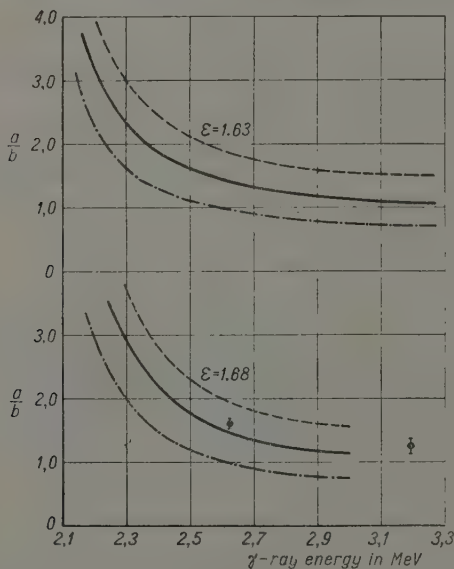


Fig. 2

protons oriented within a definite solid angle and corrected for escape effect. This assumption is valid if the concentration of hydrogen in the emulsion is uniform.

The escape correction for proton tracks being taken into account was evaluated to be about $+2\%$ and $+5\%$ for both groups of track-lengths respectively.

On the ground of these considerations and assuming the law of the angular distribution of photo-neutrons produced in ${}^9\text{Be}$ by γ -rays to be of

the form

$$\frac{d\sigma}{d\Omega} = a + b \sin^2\theta$$

the values of the ratio a/b were found to be:

$$\begin{array}{ll} \text{for } 2.62 \text{ MeV } \gamma\text{-rays} & a/b = 1.56 \pm 0.11 \\ \text{for } 3.20 \text{ MeV } \gamma\text{-rays} & a/b = 1.26 \pm 0.16. \end{array}$$

Fig. 2 represents theoretical curves from the paper of Mullin and Guth [5] evaluated under the assumption of two values 1.63 MeV and 1.68 MeV for the threshold energy for the photodisintegration of ${}^9\text{Be}$. Each set of three curves represents the dependence of the ratio a/b on the energy of γ -rays for three different theoretical treatments of the electric dipole transitions in ${}^9\text{Be}$ from the ground ${}^2\text{P}$ state to ${}^2\text{S}$ and ${}^2\text{D}$ states of positive energy, resulting in the splitting of a neutron from the remaining unstable nucleus ${}^8\text{Be}$. The middle solid curves in each set correspond to the case when the splitting of the excited level ${}^2\text{D}$ into $\text{D}_{5/2}$ and $\text{D}_{3/2}$ is not neglected.

The only value for a/b known to the authors of the present paper, which was hitherto experimentally obtained, namely $a/b = 1.22$ for the γ -ray energy 2.75 MeV, is clearly close to the middle curve. Our result for the γ -ray energy 2.62 MeV is also in good agreement with the middle curve for the threshold energy 1.68 MeV which is close to the present best experimental value of 1.67 MeV [7]. The same may be said of our result for γ -rays of the energy 3.20 MeV in respect to the smooth extension of these curves towards higher γ -ray energies. Two dots on the lower part of Fig. 2 represent our results.

Physical Laboratory, Jagellonian University, Cracow

REFERENCES

- [1] Chadwick J. and Goldhaber M., Proc. Roy. Soc., **A151** (1935), 479.
- [2] Goloborodko and Rosenkevich, Phys. Zeits. Sowjetunion, **11** (1937), 78.
- [3] Hamermesh B., Hamermesh M. and Wattenberg A., Phys. Rev., **76** (1949), 611.
- [4] Guth E. and Mullin C. J., Phys. Rev., **76** (1949), 234.
- [5] Mullin C. J. and Guth E., Phys. Rev., **76** (1949), 682.
- [6] Niewodniczański H. and Wielowiejska M., Bull. Acad. Pol. Sc., Cl. III, **1** (1953), 293.
- [7] Sher N., Halpern J. and Mann A. K., Phys. Rev., **84** (1951), 387.

Anodic Behaviour of Copper in Phosphoric Acid Solutions

by

J. KAMECKI and Z. ZEMBURA

Communicated by B. KAMIENSKI at the meeting of May 18, 1953

The anodic behaviour of copper has been arousing interest for some time past on account of its connection with the problems of polarization and passivity, and also with other problems important to industry, such as the refining of copper, galvanostegy with copper, the electrolytic polishing of copper, etc. Although there exist numerous publications on the subject of the anodic behaviour of copper, these usually deal with only one aspect of the phenomenon under observation. The works of Hedges [1] deal with the periodical changes of tension and density during a period of electrolysis under given conditions, and the works of Müller [2] discuss the chemical and mechanical passivity of a special type of anode, while Jaquet [3] and Lajner [4] deal in their works with the polishing of metal. In the most recent publications, research on the subject of anodic polishing and the properties of the surfaces so obtained has begun to predominate [5], [6]. In the research work here reported, we carried out more general investigations into the anodic behaviour of copper in phosphoric acid solutions of varying concentrations.

Experimental Section

I. Samples, Solutions, Apparatus and Method of Measurement

Samples of fire-refined rolled copper (Cu — 99.8%; Pb — 0.02%; Sb — 0.0038%; Bi — 0.0004%; Fe — 0.004%; As — 0.06%), and samples of the same sheet-coated electrolytically with acid or cyanide solutions were used as anodes. The surface area of the anode was about 30 cm.² and that of the cathode, also of copper, about 156 cm.². Before immersion the anode was polished with 3/0 and 4/0 papers. The anodes and cathodes were suspended vertically in the solutions, which were not stirred during

the period of electrolysis. Electrolysis was carried out in solutions of 1520, 1150, 495, 105 and 21 g. of H_3PO_4 per litre of solution.

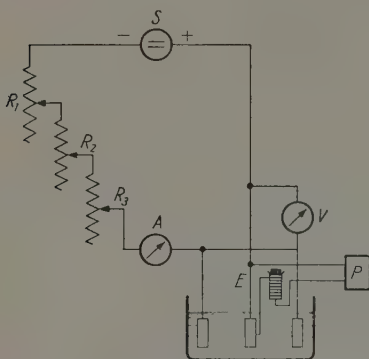


Fig. 1. Diagrams showing the equipment used for measuring anodic density, tension and anodic potential. S = selenium rectifier, R_1 , R_2 , and R_3 = rheostats with maximum resistances of 1500, 90, and 13 ohms, A = ammeter, V = voltmeter, E = saturated calomel electrode and P = compensation apparatus for measuring the anodic potential

The behaviour of the copper anodes during electrolysis was traced by measuring the relation between the density of current on the anode and the anodic potential and the tension applied to the cathode and anode, and also by calculating the degree of dependence of the tension and anodic potential on the anodic density. An illustration is given in Fig. 1 of the apparatus which was used for these measurements.

First to be measured was the anodic potential in relation to the saturated calomel electrode. The tension or density was then increased every three minutes and successive readings were made. Independently of these measurements, different phenomena arising from the anode and cathode were observed, and polishing tests under various conditions were also made.

II. Results of Measurements

The results of the measurements of the relation between the anodic density and the tension in phosphoric acid solutions of varying concentrations are shown in graph form in Fig. 2. The figures on the graphs give the concentrations of H_3PO_4 in grams per litre of solution.

The results of the measurements of the relation between the tension and the anodic density are given in Fig. 3.

Besides the measurements set out in Figs. 2 and 3, measurements were also made of the anodic potential in relation to the calomel electrode. In all cases the anodic potential was subjected to a rapid increase from the moment of passing the break in both types of curves. This rise ended practically from the time of reaching the last break. In Fig. 4 an example is given of a combination of the curve representing the ratio of anodic density to tension with the curve representing the change in anodic potential which arises in the case of electrolysis in solutions containing 1150 grams of H_3PO_4 per litre of solution.

During electrolysis, the anode at first became covered with a brown coat, which began to fall off when the tension or density became suffi-

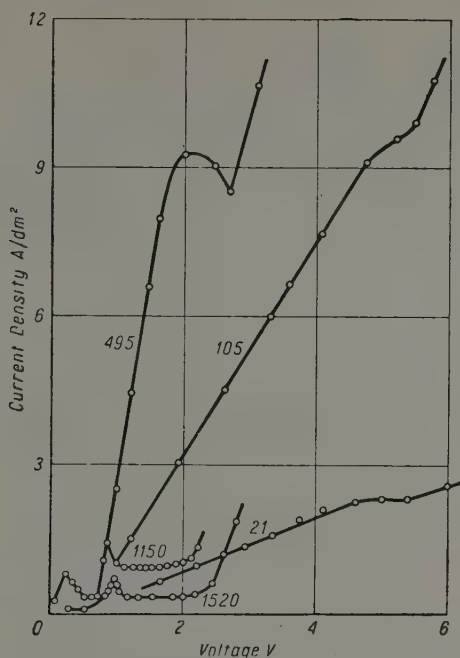


Fig. 2. Curves showing the ratio of anodic density to tension in phosphoric acid solutions

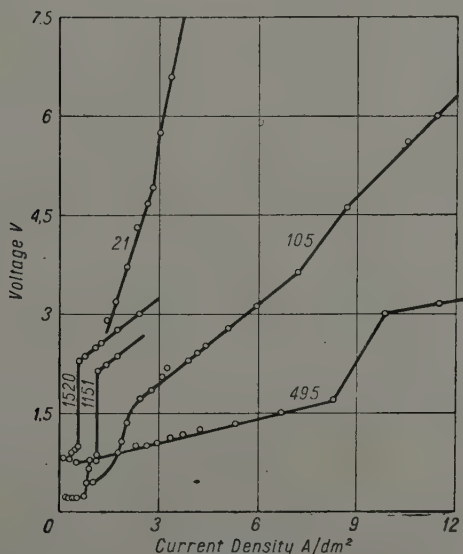


Fig. 3. Curves showing the ratio of tension to anodic density in phosphoric acid solutions

ciently high. This detachment of the coat occurred with special rapidity whenever production of oxygen began on the anode.

In all the solutions mentioned, polishing tests were also made. Polishing ensued in solutions containing from 1520 to 495 g. of H_3PO_4 per litre of solution. The best results were obtained in solutions containing 1150 g. of H_3PO_4 per litre. In these solutions the following optimum conditions of polishing were established: tension, 1.25–2 V; density of current (after at-

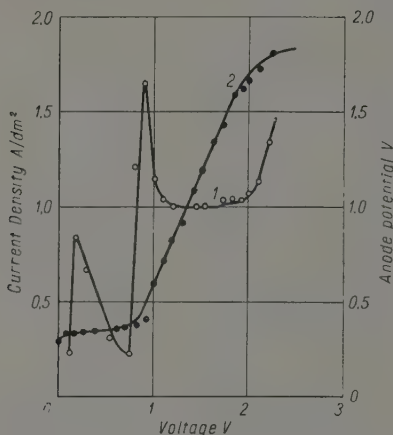


Fig. 4. Curves representing the ratio of anodic density and potential to tension in a solution containing 1150 g. of H_3PO_4 per litre of solution

taining the stationary state), from 1 to 1.2 A/dm.²; time of polishing c. 10–15 minutes. In this same solution an attempt was made to polish sheets of fire-refined copper, previously polished with 3/0 and 4/0 papers, sheets of the same material, but unpolished (coated with oxides), and also galvanically coppered coats obtained from cyanide or acid solutions. In the first case completely satisfactory results were obtained, in the second polishing ensued but left the original unevennesses and scratches, in the third polishing ensued without any hindrance, and in the last the surface became shiny, although to a lesser degree than in the previous case.

III. Discussion and Results

1. The curves of the dependence of tension on density show two breaks, (especially distinct in more concentrated solutions). The break follows upon reaching the critical density, of which the degree depends on the concentration of the solution (0.5 A/dm.² in a solution containing 1520 g.; 1.12 A/dm.² in a solution containing 1150 g.; 8.5 A/dm.² in a solution containing 495 g.

of H_3PO_4 per litre of solution). Passivity of the anode accompanies these breaks when the anodic potential rises to values of c. 0.5—1.8 V.

2. The curves of the dependence of the density on the tension have a characteristic saddle-like shape in those solutions in which polishing ensues. Polishing ensues during the period of constant density of current, which in turn depends on the degree of concentration of the solution (0.3, 1.0 and 9 A/dm.² for the first three more concentrated solutions). Since it is during this period that the passivity of the anodes appears (signalized by an increase in the anodic potential), it may be assumed that this is the period in which the increase in tension is compensated for by an increase in the electrical resistance of the coating on the anode (an increase in its thickness or a change in its porosity or chemical character).

3. The polishing tests in the solutions investigated proved that the best results were obtained in a solution containing 1150 g. of H_3PO_4 per litre of solution. This solution possessed the greatest specific conductance of all those examined.

Department of Physical Chemistry and Electrochemistry, Academy of Mining and Metallurgy, Cracow

REFERENCES

- [1] Hedges E. S., *Protective Films of Metals*, London 1937.
- [2] Müller W. J., *Die Bedeckungstheorie der Passivität der Metalle und ihre experimentale Begründung*, Berlin, 1933.
- [3] Jacquet P., *Compt. rend.* **201** (1935), 473.
— *Le polissage électrolytique des surfaces métalliques et ses applications*, 1, Paris, 1948.
- [4] Lajner W. I., *Elektroliticheskaya polirovka i travleniye metallow*, Moskwa 1947.
- [5] Batashev K. P. and Nikitin E. N., *Jurn. prikladnoy chim.* **23** (1950) 263.
- [6] Allen J. A., *Trans. Farad. Soc.* **48** (1952), 273.

The Platinum-Rhenium System

by

W. TRZEBIATOWSKI and J. BERAŁ

Communicated by W. TRZEBIATOWSKI at the meeting of October 19, 1953

Only two investigations of platinum-rhenium alloys on the platinum-rich side are known. W. Goedecke [1] deduced from measurements of electrical resistivity that alloys of platinum with 10% Re still show solid solutions. Recently A. A. Rudnicky and R. S. Poliakova [2] investigated Pt-Re alloys containing up to 17% Re by electric and hardness measurements and microscopic examination and stated that all these alloys also form solid solutions. No X-ray analysis of this system was published.

Experimental procedure

The alloys were prepared by powder methods using platinum produced from pure chloroplatinic acid and pure rhenium metal (a 99.9% product of the German Democratic Republic), thoroughly reduced with hydrogen at 1200°. The powders were mixed in desired proportions, pressed into small bars, which were placed on a corundum basis and sintered at 1500° in hydrogen during 30 hours. No substantial weight losses were found. Some parts of these samples were submitted to further annealing at 1200° (192^b) in hydrogen atmosphere and cooled by quickly removing the tube containing the samples from the furnace. Other parts were sealed in evacuated silica tubes and annealed at 900° (312^b) or 600° (690^b) and finally quenched in cold water. A second series of samples, preliminarily sintered, was melted in an arc furnace in purified argon atmosphere. Weight losses of 1–2% were observed but they did not exceed the accuracy limits of lattice constants determinations. The melted samples were subsequently annealed at 1900° in hydrogen atmosphere during 5–20 hours ($\geq 20\%$ Re), at 1700°–1800° during 2 hours ($< 20\%$ Re) and at 1500° during 30 hours (5–50% Re).

Melting points of alloys were determined for samples containing 10, 20, 30, 50% Re. These samples were placed on zirconium or rhenium bases

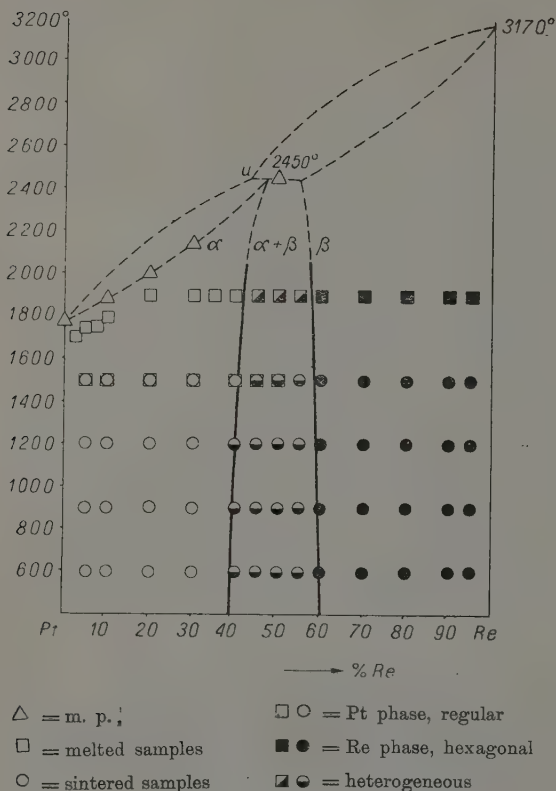


Fig. 1. Phase diagram of Pt-Re system

grams after polishing. The hardness of melted samples was determined by the Vickers method.

Results

The results of our investigations of 13 sintered and 16 melted alloys are shown on Figs. 1 and 2. Fig. 1 shows the proposed phase diagram of which all samples that were submitted to X-ray analysis are marked. It is evident that both platinum and rhenium form solid solutions of wide ranges. The heterogeneous region extends approximately from 40 to 60% Re and seems to be only very slightly dependent on temperature. The melting point of platinum rises with the rhenium content, but at $2450^\circ \pm 30^\circ$ a peritectic reaction occurs. The position of point *u* (Fig. 1) was determined microscopically. Melted alloys with rhenium content exceeding this point ($\sim 44\%$ Re) exhibit, after polishing and etching, a dendritic struc-

in carbon tubes of the Tammann furnace filled with argon. The temperature was measured by an optical pyrometer which was calibrated on melting points of palladium and platinum, applying the same experimental conditions.

X-ray analysis was carried out by powder methods. Lattice constants were determined by back reflection diagrams with Preston's focussing camera ($2r = 85.5 \text{ mm.}$). For these purposes sintered samples were powdered, placed in evacuated silica tubes, heated to the same annealing temperatures as before and again quenched in cold water. Melted samples were directly used for back reflection dia-

ture of the rhenium phase. Fig. 2 represents the values of lattice constants of samples annealed at 1500° and the Vickers hardness of melted alloys. No differences of lattice constants between melted or sintered samples were observed within the accuracy limits of $\pm 0.001 kX$. The long time of annealing at lower temperatures causes no further changes. The lattice constant of the regular platinum phase diminishes with increasing rhenium content from 3.914 to 3.889 kX at saturation composition ($\sim 42\%$ Re). The lattice constant of the rhenium phase changes irregularly. The c axis of the hexagonal rhenium cell ($a = 2.756$, $c = 4.449 kX$, $c/a = 1.614$) diminishes with increasing platinum content, but the a axis expands res-

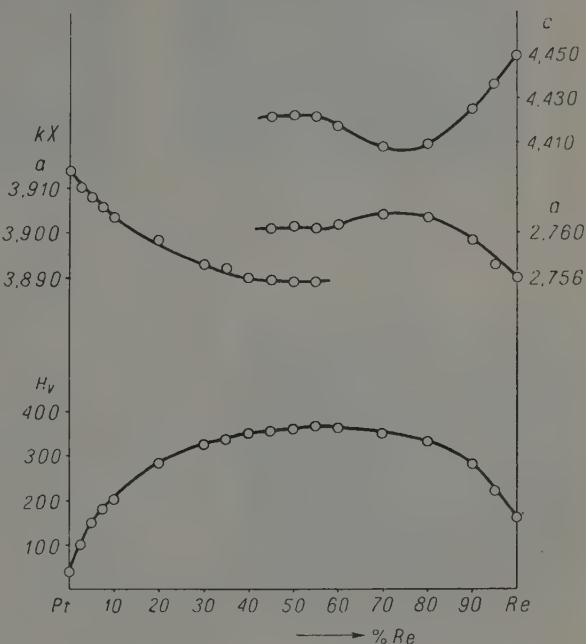


Fig. 2. Lattice constants and hardness of Pt-Re system

pectively. In the range of 70–80% Re there exist a flat minimum and maximum respectively of the lattice constants. The cell dimensions of the saturated rhenium phase ($\sim 58\%$ Re) are: $a = 2.761$, $c = 4.422 kX$, $c/a = 1.602$.

For both platinum and rhenium phases the cell volume contracts with increasing content of the second component, although rhenium offers a smaller atomic diameter than platinum. Other structural changes at lower temperatures are not evident. The hardness of platinum rises exceedingly with the rhenium content, forming a flat maximum at the heterogeneous region (Fig. 2).

Alloys containing 70 and 80% Re are particularly brittle. Microscopic examination of prepared sections confirms the results of the X-ray analysis. Fig. 3 (40% Re) shows

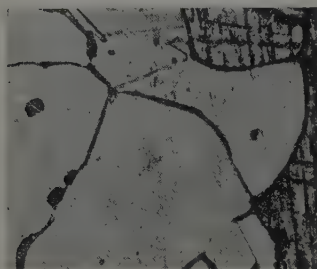


Fig. 3. 40% Re $\times 140$ -etched with aqua regia

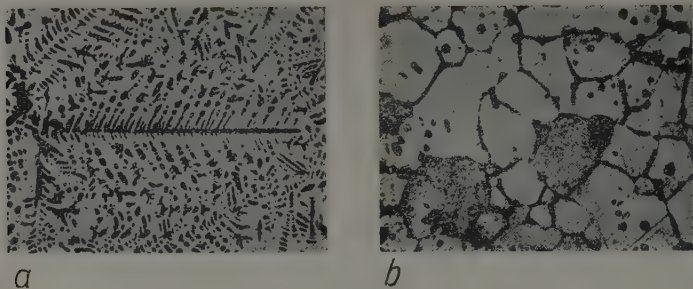


Fig. 4 a, b. 50% Re $\times 140$ -etched with n HNO₃

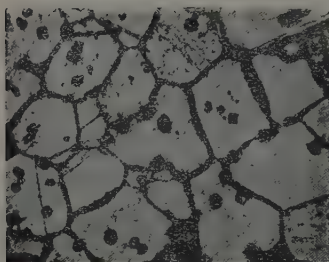


Fig. 5. 60% Re $\times 140$ -etched with n HNO₃

a homogeneous solid solution of rhenium in platinum. Fig. 4a (50% Re) shows the primary dendritic structure of the rhenium phase, as obtained by melting. This dendritic form is etched even by diluted nitric acid, contrary to the platinum phase which is attacked by *aqua regia* only. Fig. 4b represents the same sample but after annealing at 1900° for 8 hours. The dendritic structure disappears, but it is not possible to distinguish clearly by etching either of these phases, so distinctly observed in powder diagrams. Fig. 5 (60% Re) shows the homogeneous rhenium phase after annealing at 1900° for 12 hours.

We are indebted to Dr. F. Pollak for some preliminary experiments during 1941.

Department of Inorganic Chemistry, Institute of Technology, Wrocław

REFERENCES

- [1] Goedecke W., *Festschrift zum 50-jährigen Bestehen der Platinschmelze*, G. Siebert-Hanan, 1931, p. 72.
- [2] Rudnický A. A., Poliakova R. S., *Izv. Sektora Platini i Drugih Blagorodnih Metallov*, **27** (1952), 223.

Some Crystallochemical Aspects of Glauconite

by

K. SMULIKOWSKI

Communicated by J. SAMSONOWICZ at the meeting of May 18, 1953

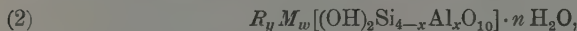
Glauconite is a mineral which is of great interest both to mineralogists and to geologists because of its wide occurrence in sedimentary formations of every geological age, from the Lower Cambrian up to the present time. The author of the present paper has been interested in glauconite from the beginning of his scientific work in mineralogy and petrology, and he has published two papers on this subject, one in 1924 and one in 1935 [1], [2]. From the time of the latter publication many important contributions to our knowledge of this mineral have appeared in world scientific literature and new light has been thrown upon many problems concerning its nature, composition and origin. An attempt has now been made by the author to synthesize all his own and foreign observations, in order to present a better and more universal picture of the constitution of glauconite and other minerals closely related to it.

The first important step in the investigation of the crystallochemical constitution of glauconite was made by Gruner [3] who, as a result of roentgenographical analysis, was able to state that glauconite, in spite of its amorphous appearance, has a fairly regular crystal-lattice structure of the same type as all micas. This is the so-called "layer structure" consisting of an alternate disposition in parallel planes of analogous ions or ion-complexes. The layers which are provided with a negative charge, $[(OH)_2Si_4O_{10}]^{-6}$, are in pairs coupled by a layer of medium cations neutralizing their charge. The coordination scheme of oxygen against silicon in the anion-layer is tetrahedral and against the medium cations octahedral; therefore the former are named "tetrahedral layers" and the latter "octahedral" ones. Tetravalent silicon in tetrahedral layers may be substituted in part by trivalent aluminium and then additional negative charges arise, rendering possible the inclusion of large cations in the intervals between the triple layers (so-called "interlayer cations" — chiefly potassium). The author demonstrated the model of the lattice structure of white mica (muscovite) and explained its

modern crystallochemical formula



This formula can serve as a prototype for the formulae of other more complicated micas, including glauconite, with the possible supplement of neutral water molecules which may be absorbed within the layer structure. Such a generalized formula may be written as follows:



where R represents cations with great ionic radii — chiefly K, accessorially Na, Rb, Ca, Sr and Ba; M represents cations with medium ionic radii — chiefly Al, Fe^{III} , Mg, Fe^{II} , accessorially Ti^{IV} , Li^I and Mn^{II} .

The indices y , x and w may vary within rather narrow limits. They are obviously very closely connected with each other, because the structure must be exactly balanced in the electrostatical sense. Their mutual dependence may be expressed by the following equation:

$$(3) \quad y + c = x + z + 3u,$$

in which c = sum of large bivalent interlayer cations

$$(c = Ca + Sr + Ba)$$

z = sum of medium cations M of a valence different from 3

$$(z = Mg^{+2} + Fe^{+2} - Ti^{+4} + 2 Li^{+1})$$

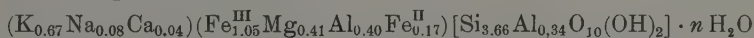
$u = 2 - w$, may be either positive, if $w < 2$, or negative, if $w > 2$

In pure muscovite which corresponds exactly to formula (1) these indices are quite simple: $y = x = 1$, $w = 2$; z, c, u equal zero. In the majority of rock-forming muscovites, showing small deviations from the theoretical formula, the indices are subject to some unimportant changes, but in the other minerals of the mica group grave complications occur, causing greater variations of all indices — a phenomenon quite characteristic for each mineral. This must also be true for glauconite which, according to Gruner, belongs to the mica group. Whatever complication may appear, the conditions of formula (2) and equation (3) must be fulfilled in every case.

In order to obtain a detailed picture of the crystallochemical constitution of glauconite, the author selected from the world scientific literature of the last 60 years all the chemical analyses of this mineral from different localities and geologic formations. Out of this material, 68 trustworthy analyses were selected and calculated on the cation ratios according to the general formula (2). The method of calculation was based on the principles of Hendricks and Ross [4] and of Harvey [5]. In order to determine the crystallochemical position of glauconite in relation to the whole mica group, 67 analyses of different minerals of this group, such as muscovite, biotite, phengite, sericite, hydromuscovite, illite and pholidoides were also selected and calculated on the same basis.

A crystallochemical comparison of all these mica minerals and especially the determination of the chemical variability of glauconite were greatly facilitated by the application of some graphical methods. Suitable diagrams were constructed by the author, who used as graphical coordinates the indices of equation (3) and some cation ratios among the octahedral cation group *M*. The conclusions of this crystallochemical discussion may be summarized as follows:

1. In typical and pure glauconites the index *x* varies between 0.1 and 0.7 but usually amounts to 0.3–0.4. The index *y* oscillates between 0.5 and 1.0, most frequently, however, between 0.7 and 0.8. In the octahedral layer the proportion of ferric iron usually amounts to 1.0–1.1, that of aluminium to 0.3–0.4; glauconites showing a preponderance of Al over Fe^{III}, i. e. so-called "aluminous glauconites" are relatively rare. The statistical average composition of typical glauconite may be illustrated by the following ratios in the general formula:



2. Pholidoides which are common micaceous components of marine marls, according to the author's definition of the term [2], and illites found by Grim and his co-workers in many slates [6], are quite separate mineral kinds, differing in their crystallochemical proportions. In this respect illites are rather similar to muscovite, pholidoides to phengite and glauconite.

3. Sericites do not constitute any separate mica mineral, but correspond either to illite, or to true muscovite, or to phengite. The name sericite designates only the microlamellar development of any light mica.

4. A simplified picture of the crystallochemical properties of different members of the mica group may be presented in the following schemes:

Glauconites	$x \ll y \leq 1$	$w \cong 2$	$Al^6 < Fe^{III}$
Pholidoides	$x < y < 1$	$w \geq 2$	$Al^6 \gg Fe^{III}$
Illites	$x \cong y < 1$	$w > 2$	$Al^6 \gg Fe^{III}$
Phengites	$x < y \cong 1$	$w = 2$	$Al^6 \gg Fe^{III}$
Muscovites	$x \cong y \cong 1$	$w = 2$	
Hydromuscovites	$x > y < 1$	$w > 2$	
Biotites	$x > y \cong 1$	$w = 2.5-2.8$	

5. The chemical variability of glauconite was shown to be dependent to some extent on the geological age and petrographic type of the glauconite-bearing sediment. This was demonstrated by a comparison of average compositions which had been calculated separately for recent, Tertiary, Cretaceous, Jurassic, Early-Palaeozoic glauconites and for glauconites from sands, sandstones, marls and limestones. Though the differences between these averages are rather slight, they are in every case systematic and consistent:

a) the older the glauconite, the smaller the prevalence of ferric iron over aluminium in the octahedral layer and the greater the proportion of interlayer cations;

b) the higher the lime content of the sediment, the smaller the prevalence of Fe^{III} over Al in the octahedral layer, and the greater the proportion of interlayer cations.

These conclusions are obviously of prime importance for geology and geochemistry. In their present state, however, they cannot be proved as indubitable. The systematic differentiation of the chemical composition of glauconite with increasing geological age and with increasing lime content of the sediment shows nearly identical trends and the role of these two quite different factors cannot be clearly separated. The identity of both trends may have been caused through an accidental selection of glauconite analyses in mineralogical literature.

Another problem to be considered in the crystallochemistry of glauconite is its water content. Two different kinds of water may be distinguished here:

1. the water of constitution taking place in an ionic condition as $(\text{OH})^{-1}$ in the tetrahedral layers of the glauconite structure. It may be expelled therefrom at higher temperatures, from 400° — 420° upwards;

2. adsorbed water in neutral molecules is contained in glauconite in variable amounts, depending on the humidity of the atmosphere (partial tension of water vapour). It may be partially expelled by drying in an ordinary temperature in exsiccators filled with concentrated sulphuric acid and afterwards regained in a humid atmosphere. It may be continually lost by heating from 40° up to 180°C . To exceed this temperature causes discoloration of the glauconite from green to rubigenous and then the water cannot return into the glauconite structure.

This indicates an alteration of the lattice structure and the closing of some interlayer spaces. From this the author is inclined to draw the conclusion that the adsorbed water is enclosed in the interlayer spaces, which must be wider and not so closely filled with interlayer cations as in true micas. Thus it would seem that the glauconite structure is intermediate between the mica type and the montmorillonite type of layer structure and this could explain the marked ability of glauconite to adsorb many organic or inorganic fluids.

REFERENCES

- [1] Smulikowski K., *Sur la glauconie*, Kosmos, **49**, Lwów (1924), 502—554.
- [2] Smulikowski K., *Skolite, un nouveau minéral du groupe de glauconie*, Arch. Minér. **12** (1936), 144—180.
- [3] Gruner J. W., *The Structural Relationship of Glauconite and Mica*, Amer. Miner., **20** (1935), 699—713.
- [4] Hendricks S. B. & Ross C. S., *Chemical Composition and Genesis of Glauconite and Celadonite*, Amer. Miner., **26** (1941), 683—708.
- [5] Harvey C. O., *Some Notes on the Calculation of Molecular Formulae for Glauconite*, Amer. Miner., **28** (1943), 541.
- [6] Grim R. E., Bray R. H. and Bradley W. F., *The Mica in argillaceous sediments*, Amer. Miner. **22** (1937), 813—829.

The Apatite-Nephelite Intrusions of the Khibina Tundras and their Origin

by

S. J. THUGUTT

Presented on July 16, 1953

Abstract: The origin of apatite-nephelite intrusions in the Khibina tundras has so far been associated with the differentiation of iolite-urtite magma. The author's opinion is that they are assimilation products of long-time accumulations of arctic cold-water life, killed by a warm current coming from the south.

According to the description of A. E. Fersmann [1], two mountain units rise up to an altitude of 1200 metres above sea level in the middle of the Kola peninsula, far north of the polar circle (between 67° 35' and 67° 55' northern latitude), from among the boggy lowlands and lakes of the taiga. They are the Khibina and the Lovoziero tundras, extending over an area of 1600 square kilometres and constituting the greatest outcrop of alkaline plutonic rocks in the whole world. The waters of the great lakes Umbozero, Lovoziero and Imandra surround the slopes of these ancient volcanic hearths, which were discovered by W. Ramsay in 1889 [2], [3] and have been investigated in detail by Soviet scientists since 1920 [4], [5].

Their history goes back to the late Silurian period, during which palaeozoic gneisses and a number of gabbro-diorite rocks, as well as the sands and arcoses overlying them, were disrupted by a series of alkaline intrusions. Coarse-grained nephelite-syenite intrusions, called *Khibinites*, are among the earliest of these. After them came fine-grained intrusions of a similar composition, separating the belt of Khibinites from the foyaite core by an arc open towards the East. The third period was characterised by an eruption of iolite-urtite magma from which the strange apatite-nephelite rock emerged at a later period. In the last pneumatolitic stage of the volcanic processes the fissures of the cooled and fractured magma were filled in by numerous pegmatites.

A. E. Fersmann is of the opinion [6] that the process of formation of the primary magma is not clear and can in no way be explained by the theory of limestone assimilation upheld by V. J. Vlodavec [7], after Daly,

even though there are no limestone deposits present in those regions. The mechanism of the Khibinite intrusion was of an explosive character. The pulsating magma, having lifted a cover composed of archaic rocks, found an exit in craters later destroyed by erosion. Saturation of the rock by apatite was slight in the magmatic stage. The principal mass of the apatite-nephelite formation arose only as a result of a differentiation of iolite-urtite magma.

Considering that this magma contains 2.15% P_2O_5 and 8.9% CaO, we are led to believe that the conception is based on an insufficiently justified differentiation; namely, 30 to 70% apatite and 25 to 40% nephelite were found in this apatite-nephelite rock. Its total quantity amounts to two thousand million tons, including 465 million tons of blotty ore containing 23 to 29% P_2O_5 . How can it, therefore, be possible that such a mass of apatite should have been produced by magmatic segregation only?

The origin of this mass of phosphorus in the Khibina rocks must be found elsewhere. According to B. Kupletskaya and A. Polkanov [8] the sea which now surrounds the Kola peninsula extended in the past as far as lake Imandra, which now lies in the middle of the peninsula. Hence it follows that the warm waters coming from the south must have brought destruction to the cold-water fauna of the Arctic Sea on reaching the foot of the Khibina massif. Here died the plankton, that is now represented by eighty various forms, and the benthos now represented by 800 forms.

During the folding of the northern European areas in the Caledonian period, large parts emerged or were submerged in turn. The history of both the Khibina massifs is connected with those movements. Repeated intrusions of alkaline magma assimilated the old cover of metamorphosed sediments together with the huge cemetery hiding the remains of extinct sea organisms. Considering the ease with which the iolite-urtite magma penetrated even the thinnest fissures of the mountain massif, one may suppose that its temperature must have been high enough to assimilate and transform properly the material, rich in calcium and phosphorus. The syntectic product *) formed in this way is this apatite-nephelite intrusion, up to 200 metres thick in some places, unique in the world and at the same time so strange.

Nowhere besides Florida, Algiers and Tunis [10], do phosphorite deposits, formed by generally known biochemical processes, reach the proportions found in the Khibina tundra. The oölite beds of phosphorite shales in the River Mountains of Wyoming [11] are 7 to 12 metres thick and contain 45 to 60% calcium phosphate; the phosphorite oölites of Melrose, Montana, [12] form a deposit of 3 to 5 metres in thickness with a content of 20 to 65% calcium phosphate; the Cenomanian phosphorites of Podolia on the

*) Under the name *Syntexis* introduced in petrology by Leewinson-Lessing F. J., we should understand the phenomenon of absorption and transformation of solid material by fluid magma [9].

river Dniestr [13] have a thickness of only 0.5 metre. On the other hand the deposit of Khibina material is evaluated at hundreds of millions of tons.

Independently of these deposits there appear in the Khibina many rare minerals containing titanium, zirconium and niobium [14]. It is possible that these elements were concentrated by biochemical processes of lower organisms, then assimilated by the magma and later carried back to its surface by means of such factors as chlorine and fluorine.

It is characteristic that here, as in Greenland and Langensundsfjord, titanium seems to prevail decidedly over zirconium.

REFERENCES

- [1] Fersmann A. E., *Priroda*, No. 5 (1929), 379—403.
- [2] Ramsay W., *Fennia*, **3** (1890), 1—52.
- [3] Ramsay W. and Hackmann V., *Fennia*, **11** (1894), 1—225; **15** (1899), 1—15.
- [4] Lomonosovski Institut Akad. Nauk., Moskva, (1937), 1—563.
- [5] Richter G., *Priroda*, No. 1 (1929), 87.
- [6] Fersmann A. E., *Geochemische Migration der Elemente und deren wissenschaftliche Bedeutung, erläutert an Chibina-Tundren*, **1** (1929), 36; Lomonosovski Inst. Akad. Nauk., Moskva, (1937), 419—440.
- [7] Vlodavec V. J., *Trudy Arkticheskovo Instituta*, **23** (1935), 5—58.
- [8] Deringin K. M., *Priroda*, No. 9 (1922), 791—806.
- [9] Scheumann K. H., *Tscherm. Min. Petr. Mitt.*, **48** (1936), 412.
- [10] Roussel J., *C. R. Acad. Sc. Paris*, **151** (1910), 600.
- [11] Condit D. D., *U. S. Geol. Survey Bull.*, **764** (1924), 39.
- [12] Richards R. W. and Pardes J. T., *U. S. Geol. Survey Bull.*, 780 (1925), 1—81; *Ref. N. J. f. Min.*, **1** (1926), 300.
- [13] Tokarski J., *Tscherm. Min. Petr. Mitt.*, **38** (1925), 599.
- [14] Fersmann A. E., *Minerals from the Kola Peninsula*, *Amer. Mineralogist*, **11** (1926), 289—299; *Ref. N. J. f. Min.*, **2** (1927), 38; *Mineral Assoziationen in Khibinsky und Lovosersky Tundren*, *Bull. Ac. Sc. R. S. F. S. R.*, (1923), 65—80.

